UDC 539.3

INFLUENCE OF PHYSICAL AND GEOMETRICAL NONLINEARITY ON THE VALUE OF THE UPPER CRITICAL PRESSURE IN BUCKLING OF A HOLLOW SPHERE

PMM Vol.42, № 3, 1978, pp.504-510

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(Received September 7,1977)

The framework of the Murnaghan model of a nonlinearly elastic body is used to investigate the problem of stability of a closed sphere acted upon by hydrostatic pressure. The theory of small deformations of an elastic body superimposed on a finite deformation is used [1]. The initial stress-strain state of equilibrium of the sphere is assumed to be centrally symmetric, and the neighboring state of equilibrium to be axisymmetric.

The conditions of bifurcation of equilibrium lead to an eigenvalue problem with a nonlinear entry of the parameter. A solution of this problem is obtained by numerical methods for spheres of varying thickness, with various constants appearing in the equation of state taken into account. Several variants of the equations of neutral equilibrium are compared, depending on the accuracy of solution of the initial problem. All this makes possible the investigation of the influence of the physical and geometrical nonlinearity on the value of the critical pressure to be carried out.

1. Let us consider three equilibrium states of an elastic body. The initial state in the volume v, the first deformed state caused by the initial load (V is the volume and O denotes the surface), and the second stress-strain state (V^* denotes the volume and O^* the surface) associated with the first state. We also have

$$\mathbf{R}^* = \mathbf{R}(\mathbf{r}) + \eta \mathbf{w}(\mathbf{r})$$

where $\mathbf{r}$, $\mathbf{R}$ and $\mathbf{R}^*$ are the radius vectors of a point of the medium in v, V and V^* respectively; η is a small parameter signifying the smallness of the additional displacement $\eta\mathbf{w}$ (in what follows, we shall neglect the terms containing η^2). We shall further assume that mass forces are absent. In this case the equations of equilibrium for the additional deformations within the volume and on the surface, can be written in the form [2]

$$\nabla'\cdot\Theta = 0, \quad \mathbf{N}\cdot\Theta = \mathbf{f}, \quad \nabla' = \nabla\mathbf{R}^{-1}\cdot\nabla \tag{1.1}$$

$$\Theta = \frac{1}{\eta}(T^* - T) + \vartheta T - \nabla'\mathbf{w}^T\cdot T$$

($\mathbf{f}$ denotes the additional surface force).
Here ∇' is the del-operator in the metric of the volume V connected with the del-operator ∇ (referred to the undeformed state), $\mathbf{N}$ is the normal vector, T and T^* are the Cauchy stress tensors in the volumes V and V^*, and ϑ is the first invariant of the linear tensor of additional deformation.

The equation of state for the Murnaghan material is given in the Finger form by

$$T = \frac{1}{4\sqrt{I_3}}\{[-6\lambda - 4\mu + 9l + n + (2\lambda - 6l + 2m - n)I_1 + lI_1^2 - 2mI_2]M + (4\mu - 6m + n + 2mI_1)M^2 + nI_3E\} \tag{1.2}$$

(E is the unit tensor).
Here $M = \nabla\mathbf{R}^T\cdot\nabla\mathbf{R}$ is the Finger measure of deformation, I_1, I_2 and I_3 are its principal invariants; λ, μ, l, m and n are the moduli of elasticity. The relation (1.2) can be obtained using the expression for the specific potential energy of deformation in the Murnaghan form, and relations connecting the invariants of the measure with the invariants of the finite deformation tensor [2]. Using the relations (1.1) and (1.2) we obtain

$$\Theta = \frac{1}{\sqrt{I_3}}\Big\{\Big[\frac{-6\lambda - 4\mu + 9l + n}{4} + \frac{2\lambda - 6l + 2m - n}{4}I_1 + \frac{l}{4}I_1^2 - \frac{m}{2}I_2\Big]\nabla\mathbf{R}^T\cdot\nabla\mathbf{w} + \Big[\frac{2\lambda - 6l + 2m - n}{2}M + (l - m)I_1M + mM^2\Big]\nabla\mathbf{R}^T\cdot\cdot\nabla\mathbf{w} + m\nabla\mathbf{R}^T\cdot\nabla\mathbf{R}\cdot\nabla\mathbf{R}^T\cdot\cdot\nabla\mathbf{w}M + \Big(\frac{4\mu - 6m + n}{2} + \frac{m}{2}I_1\Big)(M\cdot\nabla\mathbf{R}^T\cdot\nabla\mathbf{w} + M\cdot\nabla\mathbf{w}^T\cdot\nabla\mathbf{R} + \nabla\mathbf{R}^T\cdot\nabla\mathbf{w}\cdot M) + \frac{n}{4}I_3(2\nabla\mathbf{w}^T\cdot\cdot\nabla\mathbf{R}^{-T}E - \nabla\mathbf{w}^T\cdot\nabla\mathbf{R}^{-T})\Big\} \tag{1.3}$$

2. We introduce the spherical coordinates r, φ and λ and consider a centrally symmetric deformation of a hollow sphere the outer surface of which is acted upon by a uniformly distributed "follow-up" pressure p (the inner surface is load-free). Then the first deformed state can be described by the following equations:

$$\mathbf{R} = [r + u(r)]\mathbf{e}_r$$
$$\nabla\mathbf{R} = \nabla\mathbf{R}^T = M^{1/2} = a\mathbf{e}_r\mathbf{e}_r + b(\mathbf{e}_\varphi\mathbf{e}_\varphi + \mathbf{e}_\lambda\mathbf{e}_\lambda) \tag{2.1}$$
$$a = 1 + \frac{du}{dr}, \quad b = 1 + \frac{u}{r}, \quad \mathbf{f} = -p(\vartheta\mathbf{N} + \mathbf{N}\cdot\nabla'\mathbf{w}^T)$$

Here $u(r)$ is the radial displacement, $\mathbf{f}$ is the additional surface force vector and $\mathbf{e}_r$, $\mathbf{e}_\varphi$ and $\mathbf{e}_\lambda$ are the basis vectors coinciding with the unit vectors of the spherical coordinate system.

The second equation of (2.1) yields the principal invariants of the deformation measure

$$I_1 = a^2 + 2b^2, \quad I_2 = 2a^2b^2 + b^4, \quad I_3 = a^2b^4 \tag{2.2}$$

The adjacent forms of equilibrium situated close to the first stress-strain state are assumed to be axisymmetric, i.e.

$$\mathbf{w} = v(r, \varphi)\mathbf{e}_r + w(r, \varphi)\mathbf{e}_\varphi$$

Using the second equation of (2.1), (2.2) and (1.3), we can reduce the expression for the tensor Θ to the form

$$\Theta = \Big(A_1v + A_2\frac{\partial v}{\partial r} + A_3\frac{\partial w}{\partial\varphi} + A_4w\operatorname{ctg}\varphi\Big)\mathbf{e}_r\mathbf{e}_r + \tag{2.3}$$

$$+\left(B_1 v + B_2 \frac{\partial v}{\partial r} + B_3 \frac{\partial w}{\partial \varphi} + B_4 w \operatorname{ctg} \varphi\right)(\mathbf{e}_\varphi \mathbf{e}_\varphi + \mathbf{e}_\lambda \mathbf{e}_\lambda) +$$

$$\left(C_1 w + C_2 \frac{\partial w}{\partial r} + C_3 \frac{\partial v}{\partial \varphi}\right)\mathbf{e}_\varphi \mathbf{e}_r + \left(D_1 w + D_2 \frac{\partial w}{\partial r} + D_3 \frac{\partial v}{\partial \varphi}\right)\mathbf{e}_r \mathbf{e}_\varphi$$

The coefficients accompanying the unknowns are determined by the formulas

$$rA_1 = \frac{a}{b}\left[\lambda + l(a^2 + 2b^2 - 3) + m(1 - b^2) + \frac{n}{2}(b^2 - 1)\right] \tag{2.4}$$

$$A_2 = \frac{1}{8b^2}[2\lambda(3a^2 + 2b^2 - 3) + 4\mu(3a^2 - 1) + l(5a^4 + 4b^4 + 12a^2b^2 - 18a^2 - 12) + 2m(5a^4 + b^4 - 6a^2) + n(b^4 - 2b^2 + 1)]$$

$$rB_1 = \frac{1}{8ab}[2\lambda(a^2 + 6b^2 - 3) + 4\mu(3b^2 - 1) + l(a^4 + 20b^4 + 12a^2b^2 - 6a^2 - 36b^2 + 9) + 2m(5b^4 - 3a^2b^2 - 3b^2 + a^2) + n(1 - a^2 - 3b^2 + 3a^2b^2)]$$

$$B_2 = {}^1/_2[\lambda + l(a^2 + 2b^2 - 3) + (m - n)(1 - b^2)]$$

$$rB_3 = \frac{1}{8ab}[2\lambda(4b^2 + a^2 - 3) + 4\mu(3b^2 - 1) + l(9 - 6a^2 - 24b^2 + 8a^2b^2 + 12b^4 + a^4) + 2m(a^2 - a^2b^2 - 5b^2 + 5b^4) + n(1 - a^2 - b^2 + a^2b^2)]$$

$$rD_1 = -rD_3 = \frac{a}{8b}[-4\mu + 2m(3 - a^2 - 2b^2) + n(b^2 - 1)]$$

$$D_2 = \frac{1}{8b^2}[2\lambda(a^2 + 2b^2 - 3) + 4\mu(a^2 + b^2 - 1) + l(a^4 + 4b^4 + 4a^2b^2 - 6a^2 - 12b^2 + 9) + 2m(a^4 + b^4 + a^2b^2 - 2a^2 - b^2) + n(1 - b^2)], \quad A_3 = A_4 = {}^1/_2 A_1, \quad B_4 = B_1 - B_3$$

$$C_1 = -C_3 = -\frac{b}{ra}D_2, \quad C_2 = -\frac{b}{a}rD_1$$

Substituting the expressions (2.3) into the first condition of equilibrium (1.1), we arrive at the following system of two differential equations:

$$\frac{\partial}{\partial r}\left[A_1 v + A_2 \frac{\partial v}{\partial r} + \frac{A_1}{2}\left(\frac{\partial w}{\partial \varphi} + w \operatorname{ctg} \varphi\right)\right] + \left\{\frac{2}{r}\left[A_1 v + A_2 \frac{\partial v}{\partial r} + \frac{A_1}{2}\left(\frac{\partial w}{\partial \varphi} + w \operatorname{ctg} \varphi\right)\right] - \frac{1}{r}\left[2B_1 v + 2B_2 \frac{\partial v}{\partial r} + B_1\left(\frac{\partial w}{\partial \varphi} + w \operatorname{ctg} \varphi\right)\right] + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(C_1 w + C_2 \frac{\partial w}{\partial r} - C_1 \frac{\partial v}{\partial \varphi}\right) + \frac{1}{r}\left(C_1 w + C_2 \frac{\partial w}{\partial r} - C_1 \frac{\partial v}{\partial \varphi}\right)\operatorname{ctg} \varphi\right\}\frac{a}{b} = 0$$

$$\frac{\partial}{\partial r}\left(D_1 w + D_2 \frac{\partial w}{\partial r} - D_1 \frac{\partial v}{\partial \varphi}\right) + \left[\frac{2}{r}\left(D_1 w + D_2 \frac{\partial w}{\partial r} - D_1 \frac{\partial v}{\partial \varphi}\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(B_1 v + B_2 \frac{\partial v}{\partial r} + B_3 \frac{\partial w}{\partial \varphi} + B_4 w \operatorname{ctg} \varphi\right) + \frac{1}{r}(B_3 - B_4)\left(\frac{\partial w}{\partial \varphi} - w \operatorname{ctg} \varphi\right)\operatorname{ctg} \varphi + \frac{1}{r}\left(C_1 w + C_2 \frac{\partial w}{\partial r} - C_1 \frac{\partial v}{\partial \varphi}\right)\right]\frac{a}{b} = 0$$

and we seek the solution of this system in the form

$$v = X_k(r) P_k(\cos\varphi), \quad w = Y_k(r) \frac{dP_k(\cos\varphi)}{d(\cos\varphi)} \sin\varphi \tag{2.5}$$

where P_k (cos φ) is the k-th order Legendre polynomial. On substituting (2.5) we see that the variables separate. As a result, we obtain a system of ordinary differential equations with variable coefficients (a prime denotes the derivative with respect to the radial coordinate)

$$\frac{1}{r}\left\{\left[A_1\left(X_k + \frac{k^2+k}{2} Y_k\right) + A_2X_k'\right]r^2\right\}' - \frac{a}{b}\left[2B_1\left(X_k + \frac{k^2+k}{2} Y_k\right) + 2B_2X_k' - (k^2+k)\,C_1(X_k+Y_k) - (k^2+k)\,C_2Y_k'\right] = 0 \tag{2.6}$$

$$\frac{1}{r}\{[D_1(X_k+Y_k) + D_2Y_k']\,r^2\}' + \frac{a}{b}\left[(C_1+2B_3-B_1)(X_k+Y_k) + C_2Y_k' - 2B_3\left(X_k + \frac{k^2+k}{2} Y_k\right) - B_2X_k'\right] = 0$$

The second condition of equilibrium given in (1.1) on the outer $(r = r_0)$ and inner $(r = r_1)$ boundary of the sphere yields the following four relations:

$$\left(\frac{rA_1}{2} + \frac{p}{2b}\right)\left[X_k' + \frac{2}{r}\left(X_k + \frac{k^2+k}{2} Y_k\right)\right] + \left(A_2 - \frac{rA_1}{2} - \frac{p}{2b}\right)X_k' = 0 \tag{2.7}$$

$$\left(rD_1 + \frac{p}{b}\right)\left[Y_k' + \frac{1}{r}(X_k+Y_k)\right] + \left(D_2 - rD_1 - \frac{p}{2b}\right)Y_k' = 0 \qquad (r = r_0)$$

$$\frac{rA_1}{2}\left[X_k' + \frac{2}{r}\left(X_k + \frac{k^2+k}{2} Y_k\right)\right] + \left(A_2 - \frac{rA_1}{2}\right)X_k' = 0$$

$$rD_1\left[Y_k' + \frac{1}{r}(X_k+Y_k)\right] + (D_2 - rD_1)\,Y_k' = 0 \qquad (r = r_1)$$

From the formulas (2.4) for the coefficients $A_1, A_2, \ldots, C_1, C_2$ it is clear that the load p enters the equations of neutral equilibrium through the functions a and b. The homogeneous system of Eqs.(2.6) with boundary conditions (2.7) has a trivial solution $w = 0$. However, at certain values of p called the bifurcation values, non-trivial solutions corresponding to the perturbed equilibrium forms of the volume V^* also become possible. The critical pressure is found as the smallest bifurcation value of p determined by the appropriate choice of the number of nodes k of the bifurcation form. We note that the eigenvalue problem (2.6),(2.7) of determining the bifurcation load is nonlinear, since the parameter p to be determined enters the expression in a nonlinear manner (the operator determined by the system and the boundary conditions is however linear).

3. The problem of initial deformation of the sphere reduces, in the case of the Murnaghan material, to the following nonlinear boundary value problem [2]:

$$\sigma_r' + \frac{2R'}{R}(\sigma_r - \sigma_\varphi) = 0 \tag{3.1}$$

$$\sigma_r(r_0) = -p, \quad \sigma_r(r_1) = 0,$$

The stresses are determined by the equations

$$\sigma_r = \frac{a}{4b^2}[-6\lambda - 4\mu + 9l + n + (2\lambda + 4\mu - 6l - 4m)a^2 + (4\lambda - 12l + 4m - 2n)b^2 + (l + 2m)a^4 + 4la^2b^2 + (4l - 2m + n)b^4]$$

$$\sigma_r - \sigma_\varphi = \left(\frac{a}{4b^2} - \frac{1}{4a}\right)[-6\lambda - 4\mu + 9l + n + (2\lambda + 4\mu - 6l - 4m)\times (a^2 + 2b^2) + (l + 2m)a^4 + (4l + 4m)a^2b^2 + (4l + 6m)b^4 + (6m - 4\mu - n)b^2 - 2m(a^2b^2 + 2b^4)]$$

An attempt to solve the boundary value problem (3.1) analytically was without success. The displacements $u(r)$ (and therefore the functions a and b) were obtained in approximate manner. The values obtained were then used to determine the coefficients of the problem (2.6), (2.7) and for this reason the equations of neutral equilibrium differ somewhat from the exact equations, the difference depending on the accuracy of solution of the initial problem (3.1). Study of the influence of the error present in the solution of the initial problem on the value of the critical pressure is of interest.

Three variants of the equations of neutral equilibrium were investigated. In the first case the boundary value problem (3.1) was linearized and the initial displacement had the form

$$u = \frac{pr_1^3}{r_0^3 - r_1^3}\left(\frac{r}{3\lambda + 2\mu} + \frac{r_0^3}{4\mu r^2}\right) \equiv u_1 \tag{3.2}$$

In the second case the boundary value problem (3.1) was solved with the second order of smallness terms taken into account [2]. The following, more accurate expression was obtained for the displacements within the framework of the second order theory:

$$u = u_1 + \alpha r + \beta\frac{1}{r^2} + \gamma\frac{1}{r^5}$$

where the constants α, β can be obtained from the system of linear algebraic equations the first of which is

$$(3\lambda + 2\mu)\alpha - \frac{4\mu}{r_1^3}\beta = \frac{3\lambda + 10\mu}{r_1^6} - \left(\lambda + m - \frac{n}{2}\omega(r_1) + \left(\lambda - l + m - \frac{n}{2}\right)\vartheta_1^2(r_1) - (2\lambda - 2\mu + 2m - n)\vartheta_1(r_1)u_1'(r_1) - (5\mu + n)[u_1'(r_1)]^2\right.$$

and the second one is obtained by replacing r_1 and r_0. The following notation is used here:

$$\vartheta_1 = \nabla\cdot\mathbf{u}_1, \qquad \omega = E\cdot\cdot\left(\frac{\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^T}{2}\right)^2$$

$$\mathbf{u}_1 = u_1(r)\,\mathbf{e}_r, \qquad \gamma = \frac{(\lambda + 3\mu + 2m)\,p^2r_0^6}{(\lambda + 2\mu)(r_1^3 - r_0^3)^2(3\lambda + 2\mu)^2}$$

In the third case the initial problem (3.1) is integrated using the finite difference method. The equidistant nodes were used and the derivatives were approximated with the central second order differences. The resulting system of nonlinear algebraic equations was solved using the generalized Steffensen [3] method, and the choice of the initial approximation was based on the solution of the linear problem with the initial displacement (3.2). The interval (r_1, r_0) was divided into 30 to 100 subintervals depending on the thickness of the sphere. All approximate calculations were repeated twice to ensure accuracy.

The eigenvalues of the homogeneous problem (2.6), (2.7) were obtained by numerical methods. The method of discretization was used to solve the nonlinear eigenvalue problems, and its use can be justified with the help of the convergence theorem given in [4].

The system (2.6) can be written in normal form, which is better suited for the subsequent computations, by passing to new variables and introducing the functions y_j according to the relations

$$y_1 = X_k, \quad y_2 = X_k', \quad y_3 = Y_k, \quad y_4 = Y_k'$$

Then the general solution of the system can be written in the form

$$\mathbf{y}(r;\, p) = \sum_{i=1}^{4} C_i \mathbf{y}_i(r;\, p) \tag{3.3}$$

where C_i are arbitrary constants and $\{\mathbf{y}_i\}$ denotes the totality of linearly independent solutions. After computing the coefficients of the system (formulas for a and b in (2.1) and formulas (2.4)), the values of the vector functions $\mathbf{y}_i(r;\, p)$ at the right end of the interval (r_1, r_0) were determined by integrating the system by the Kutta — Merson method using the following initial vectors:

$$\mathbf{y}_1(r_1;\, p) = \{0,\ k+1,\ 0,\ A_2/D_2\}$$
$$\mathbf{y}_2(r_1;\, p) = \{0,\ -k,\ 0,\ A_2/D_2\}$$
$$\mathbf{y}_3(r_1;\, p) = \{-k,\ -k^2-k,\ 1,\ k-1\}$$
$$\mathbf{y}_4(r_1;\, p) = \{k+1,\ -k^2+3k-2,\ 1,\ -k-2\}$$

The above initial conditions were chosen in accordance with the general solution of a homogeneous system, similar to (2.6) and appearing in the problem of bifurcation of equilibrium of a sphere made of semi-linear material [5]. Substitution of the general solution (3.3) into the boundary conditions (2.7) yields a homogeneous linear system of algebraic equations in C_i, $i = 1,2,3,4$. The determinant $\Delta(p)$ of this system vanishes at the bifurcation values of p, and the sign of the determinant changes on the passage through the bifurcation value. Because of this, the critical load was determined by division into halves. The numerical process of constructing the equation $\Delta(p) = 0$ and of its solution was repeated for several values of k within a sufficiently large range, and the smallest bifurcation value of p corresponding to the critical load was chosen.

4. The algorithm for the determination of critical pressures was realized on a digital computer. The results were obtained with the help of the following dimensionless quantities: relative thickness ε of the sphere and the critical value of the load parameter p_* (E is the Young's modulus)

$$\varepsilon = 2\frac{r_0 - r_1}{r_0 + r_1}, \quad p = p_* E \varepsilon^2$$

Table 1 gives the values of the critical parameter p_* and of the corresponding number k of nodes of the equilibrium bifurcation form of the sphere for various ε, for the materials the elastic properties of which are defined by the constants [6] $l = -7\cdot10^{12}$ dyn $/\mathrm{cm}^2$, $m = 0$, $n = -8.2\cdot10^{12}$ dyn $/\mathrm{cm}^2$. The following values were adopted for the Poisson's ratio and Young's modulus: $\nu = 0.272$, $E = 2\cdot10^{12}$dyn $/\mathrm{cm}^2$, and the following sets of constants l, m and n correspond to the variants 1) — 4): 1) $l = m = n = 0$, 2) $l = -7\cdot10^{12}$ dyn $/\mathrm{cm}^2$, $m = n = 0$; 3) $l = -7\cdot10^{12}$ dyn /

cm^2, $m = 0, n = -8.2 \cdot 10^{12}$ dyn/cm^2; 4) $l = m = 0, n = -8.2 \cdot 10^{12}$ dyn/cm^2.

Table 1

$\varepsilon \cdot 10^3$	1.41	12.2	100	333	$\varepsilon \cdot 10^3$	1.41	12.2	100	333
k	48	16	5	3	k	48	16	5	3
1)	1.200 1.200 1.199	1.193 1.192 1.192	1.12 1.11 1.11	0.753 0.758 0.770	3)	1.193 1.199 1.197	1.150 1.190 1.190	0.81 1.08 1.11	0.580 — —
2)	1.192 1.200 1.196	1.159 1.209 1.199	0.89 1.16 1.19	0.550 — 0.740	4)	1.196 1.198 1.197	1.123 1.178 1.190	0.98 1.07 1.03	— — —

The uppermost value in each partition of Table 1 denotes the critical value of the load parameter p_* obtained from the relations (2.6), (2.7) in which the initial displacement is taken according to the linear theory; the middle value denotes p_* corresponding to the equations written within the framework of the second order theory of elasticity, and the lower value denotes p_* corresponding to a more accurate computation of the initial stress-strain state of the sphere obtained using the finite difference method. A dash denotes the absence of the critical parameter in the corresponding interval. The variants 1) and 3) with a nonzero constant m equal [6] to $-8 \cdot 10^{12}$ dyn / cm^2 were also considered. In both cases the values of the dimensionless critical load parameter exceeded 1.2 appreciably, irrespective of the sphere thickness and the accuracy of the equations of neutral equilibrium.

Analysis of the numerical results leads to the following conclusions concerning the influence of nonlinearity on the magnitude of the upper critical pressure.

1°. In the case of thin-walled spheres the critical load p obtained from the "thick" linearized equations differs very little from the values obtained using more accurate theories whether or not the constants l and n were taken into account. The results diverge with increasing thickness of the sphere wall.

2°. The values of the dimensionless critical parameter p_* decrease with increasing relative thickness ε.

3°. The elastic Murnaghan constants with the exception of m have no significant influence on the magnitude of the upper critical pressure.

The author thanks L.M. Zubov for the interest shown.

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Translated by L.K.